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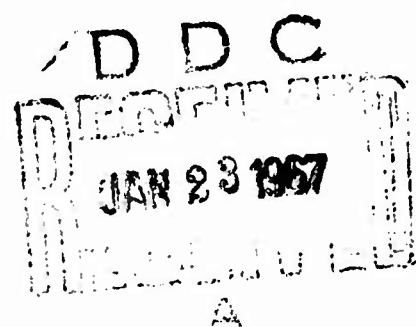
Natural Families of Periodic Orbits

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NATURAL FAMILIES OF PERIODIC ORBITS

by

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SUMMARY

In reference to any solution of a conservative
dynamical system with two degrees of freedom, Hill's equation
is generalized to encompass non-necessarily isoenergetic
displacements as well as the isoenergetic displacements
caused by a variation of a parameter.

This new variational equation is made the foundation
of a methodical procedure for continuing numerically natural
families of periodic orbits. The method consists of two steps--
an isoenergetic corrector and a tangential predictor.

Although the algorithm makes no assumption of symmetry
on the periodic orbits to be continued, special attention is
paid to the symmetric orbits, but only to show how in these
cases the method can be simplified substantially.

The subject matter of this paper is nothing else but the
elementary problem of continuing numerically analytic manifolds of
periodic orbits for a conservative dynamical system with two degrees
of freedom. Apparently for the first time, a methodical answer is
given here. Our procedure, which throughout adheres strictly
to the analytical foundations of the question, aims at keeping as
close as possible to the basic technique of Poincaré, namely the method
of analytic continuation based on Cauchy's local existence theorem.
In the classical contributions to this problem (Darwin, Stromgren,
Lemaître), the main issue is immediately obscured by the accidental
fact that, for some families, the periodicity conditions can be replaced
by symmetry conditions, and therefore the problem of correcting initial
conditions is replaced by that of adjusting boundary conditions.
Evenmore, in recent years, in spite of the many capabilities offered
by electronic computers, the numerical continuation of a manifold of
periodic orbits tended to degenerate into disreputable tricks based on
optical illusions rather than on analytical certainties. The result has
been an overabundance of numerical material whose subjective interpretation
leads to conclusions at variance with propositions firmly established by
analysis. The classical instance of such unfortunate accidents is the
still open controversy concerning a genealogy of periodic orbits
established numerically by Darwin (1897) and questioned by Poincaré (1899)
on analytical grounds.

Of course, one can only expect to find here the most elementary part of the very extensive theory of periodic orbits. A natural family of periodic orbits is defined by the local existence of power series in Painlevé's constant of integration to represent the manifold (§1). The fundamental result about a periodic orbit is an extension to non-necessarily isoenergetic normal displacements of Hill's equation which is valid only for isoenergetic variations (§2). Therefrom is derived the vital possibility of converging to a periodic orbit without leaving the energy manifold on which it lies (§4) as well a tangential predictor (§5) which expresses the "solidarity" between the phase states of a natural family on different periodic orbits of the manifold where it is deprived of singularities. Symmetric periodic orbits are given here a special treatment only because it is possible to obtain them by integration over only half an estimate of their period (§7).

After the two fundamental steps of our numerical continuation have been derived, the other sections of this paper concern what is probably the most useful concept in the theory of conservative systems, namely the isoenergetic rate of variation of the state variables with respect to a parameter. We show how such rates can be computed intrinsically here again from an extension of Hill's equation (§8). It enables us to transpose our procedure of numerical continuation to the cases when new time variables are introduced for any fixed value of the energy constant (§9).

1. NUMERICAL CONTINUATION OF A NATURAL FAMILY

Given a conservative dynamical system with two degrees of freedom

$$\mathcal{L} = \frac{1}{2}(g_{11}q_1'^2 + 2g_{12}q_1'q_2' + g_{22}q_2'^2) + f_1q_1' + f_2q_2' + U$$

one can always choose an isothermal set of coordinates (x,y) and redefine the independent variable so that the equations of motion take on the simple form

$$\ddot{x} = 2A\dot{y} + W_x, \qquad \ddot{y} = -2A\dot{x} + W_y \qquad (1)$$

where A and W are functions of the coordinates x and y (Birkhoff 1915).

The equations (1) admit the integral

$$C = 2W - (\dot{x}^2 + \dot{y}^2) \qquad (2)$$

which it will be convenient to refer to as the *energy integral*; C is an arbitrary constant of integration, to be called the *constant of energy* or also *Painlevé's* constant of integration (Chazy 1953).

Let us assume that, for the initial conditions $(x_0,y_0,\dot{x}_0,\dot{y}_0)$ at time $t = 0$, the equations (1) possess a solution

$$x(t,x_0,y_0,\dot{x}_0,\dot{y}_0), \qquad y(t,x_0,y_0,\dot{x}_0,\dot{y}_0) \qquad (3)$$

which is periodic. We denote by T_0 the period of this solution, and by C_0 the value of the energy integral along it.

We ask ourselves for what corrections Δx_0, Δy_0, $\Delta \dot{x}_0$, $\Delta \dot{y}_0$ on the initial conditions a displacement of the orbit (3) will be a periodic orbit with about the same period. Let us denote by $T_0 + \Delta T_0$ the period of this varied orbit, and by $C_0 + \Delta C_0$ its constant of energy. On expanding the periodicity conditions

$$x(T_0+\Delta T_0, x_0+\Delta x_0, y_0+\Delta y_0, \dot{x}_0+\Delta \dot{x}_0, \dot{y}_0+\Delta \dot{y}_0) = x_0 + \Delta x_0,$$

$$y(T_0+\Delta T_0, x_0+\Delta x_0, y_0+\Delta y_0, \dot{x}_0+\Delta \dot{x}_0, \dot{y}_0+\Delta \dot{y}_0) = y_0 + \Delta y_0,$$

$$\dot{x}_0(T_0+\Delta T_0, x_0+\Delta x_0, y_0+\Delta y_0, \dot{x}_0+ \dot{x}_0, \dot{y}_0+\Delta \dot{y}_0) = \dot{x}_0 + \Delta \dot{x}_0,$$

$$\dot{y}_0(T_0+\Delta T_0, x_0+\Delta x_0, y_0+\Delta y_0, \dot{x}_0+\Delta \dot{x}_0, \dot{y}_0+\Delta \dot{y}_0) = \dot{y}_0 + \Delta \dot{y}_0,$$

in power series of the corrections and on retaining only the first order terms, we come to the linear system

$$\left[\frac{\partial}{\partial x_0} x(T_0)-1\right]\Delta x_0 + \frac{\partial}{\partial y_0} x(T_0)\Delta y_0 + \frac{\partial}{\partial \dot{x}_0} x(T_0)\Delta \dot{x}_0 + \frac{\partial}{\partial \dot{y}_0} x(T_0)\Delta \dot{y}_0 + \dot{x}_0\Delta T = 0,$$

$$\frac{\partial}{\partial x_0} y(T_0)\Delta x_0 + \left[\frac{\partial}{\partial y_0} y(T_0)-1\right]\Delta y_0 + \frac{\partial}{\partial \dot{x}_0} y(T_0)\Delta x_0 + \frac{\partial}{\partial \dot{y}_0} y(T_0)\Delta \dot{y}_0 + \dot{y}_0\Delta T = 0,$$

$$\frac{\partial}{\partial x_0} \dot{x}(T_0)\Delta x_0 + \frac{\partial}{\partial y_0} \dot{x}(T_0)\Delta y_0 + \left[\frac{\partial}{\partial \dot{x}_0} \dot{x}(T_0)-1\right]\Delta \dot{x}_0 + \frac{\partial}{\partial \dot{y}_0} \dot{x}(T_0)\Delta \dot{y}_0 + \ddot{x}_0\Delta T = 0,$$

$$\frac{\partial}{\partial y_0} \dot{y}(T_0)\, x_0 + \frac{\partial}{\partial y_0} \dot{y}(T_0)\Delta y_0 + \frac{\partial}{\partial \dot{x}_0} \dot{y}(T_0)\Delta \dot{x}_0 + \left[\frac{\partial}{\partial \dot{y}_0} \dot{y}(T_0)-1\right]\Delta \dot{y}_0 + \ddot{y}_0\Delta T = 0,$$

(4)

to which we add the equation

$$\frac{\partial C}{\partial x_0} \Delta x_0 + \frac{\partial C}{\partial y_0} \Delta y_0 + \frac{\partial C}{\partial \dot{x}_0} \Delta \dot{x}_0 + \frac{\partial C}{\partial \dot{y}_0} \Delta \dot{y}_0 = \Delta C_0 \tag{5}$$

expressing that the corrections on the initial conditions result in a first order correction ΔC_0 on the constant of energy.

Let us assume that the rank of the matrix

$$\begin{pmatrix} \frac{\partial}{\partial x_0} x(T_0)-1 & \frac{\partial}{\partial y_0} x(T_0) & \frac{\partial}{\partial \dot{x}_0} x(T_0) & \frac{\partial}{\partial \dot{y}_0} x(T_0) & \dot{x}(T_0) \\[2ex] \frac{\partial}{\partial x_0} y(T_0) & \frac{\partial}{\partial y_0} y(T_0)-1 & \frac{\partial}{\partial \dot{x}_0} y(T_0) & \frac{\partial}{\partial \dot{y}_0} y(T_0) & \dot{y}(T_0) \\[2ex] \frac{\partial}{\partial x_0} \dot{x}(T_0) & \frac{\partial}{\partial y_0} \dot{x}(T_0) & \frac{\partial}{\partial \dot{x}_0} \dot{x}(T_0)-1 & \frac{\partial}{\partial \dot{y}_0} \dot{x}(T_0) & \ddot{x}(T_0) \\[2ex] \frac{\partial}{\partial x_0} \dot{y}(T_0) & \frac{\partial}{\partial y_0} \dot{y}(T_0) & \frac{\partial}{\partial \dot{x}_0} \dot{y}(T_0) & \frac{\partial}{\partial \dot{y}_0} \dot{y}(T_0)-1 & \ddot{y}(T_0) \\[2ex] \frac{\partial C}{\partial x_0} & \frac{\partial C}{\partial y_0} & \frac{\partial C}{\partial \dot{x}_0} & \frac{\partial C}{\partial \dot{y}_0} & 0 \end{pmatrix} \tag{6}$$

is equal to 4. Then on applying Poincaré's method of continuity (Siegel 1956), one can show that the corrections on the initial conditions and the period are analytic functions of ΔC_0 in the neighborhood of the initial energy constant C_0. In other words, there exist an interval $\mathcal{J}$ around C_0 and 5 analytic functions

$$X_0 = x_0 + \sum_{k \geq 1} x_{0,k}(C-C_0)^k,$$

$$Y_0 = y_0 + \sum_{k \geq 1} y_{0,k}(C-C_0)^k,$$

$$\dot{X}_0 = \dot{x}_0 + \sum_{k \geq 1} \dot{x}_{0,k}(C-C_0)^k,$$

(7)

$$\dot{Y}_0 = \dot{y}_0 + \sum_{k \geq 1} \dot{y}_{0,k}(C-C_0)^k,$$

$$T = T_0 + \sum_{k \geq 1} T_{0,k}(C-C_0)^k$$

(8)

on that interval $\mathcal{I}$ such that the solutions of (1) having the initial conditions (7) are periodic with the period T as given by (8).

Such a one-parameter manifold of orbits is what Wintner (1931) calls a *natural* family; a periodic orbit which belongs to a natural family is called *singular* by Whittaker (1916).

Since the dynamical system is conservative, the conditions (7) imply that there exist two sequences $x_k(t)$, $y_k(t)$ of functions which are periodic with period T such that the series

$$X(t) = x(t) + \sum_{k \geq 1} x_k(t)(C-C_0)^k,$$

(9)

$$Y(t) = y(t) + \sum_{k \geq 1} y_k(t)(C-C_0)^k$$

represent the natural family $O(C)$ in the neighborhood of its element $O(C_0)$ given by (3).

A natural family of periodic orbits defines in the phase space a two dimensional torus upon which, given a convenient definition of the initial point on each orbit, the time t and the Painlevé constant C constitute a system of analytic coordinates.

Solving the problem of continuing *analytically* the torus $O(C)$ from the initial orbit $O(C_0)$ means determining the series (8) and the time dependent coefficients x_k and y_k in the series (9). In a few simple cases, these functions can be determined. But in general the solution, in this most complete sense, is not feasible. To start with, the generating solution $O(C_0)$ is usually obtained by integrating numerically the equations of motion (1), so that the very first coefficients $x(t)$ and $y(t)$ are obtained only in the form of tables where only a finite number of points along $O(C_0)$ are entered. And even when the initial orbit is expressed in a somewhat more explicit way as a function of the time, it is most often not possible to find in the same explicit way even the time functions $x_1(t)$ and $y_1(t)$ in the expansions (9).

Since the analytic continuation of a natural family is generally intractable, it is quite important to settle upon methods of *numerical continuation*. The difference between analytical continuation and numerical continuation can be sensed most distinctly in geometric terms.

We think of the equations (9) as defining the continuous deformation of a periodic orbit on the torus $O(C)$. Starting from the location

$$\{x(t;C_0),y(t;C_0),\dot{x}(t;C_0),\dot{y}(t;C_0): \ 0 \leq t \leq T_0\}$$

in the phase space, the orbit will stretch and twist under the deformation so as to occupy, when the constant of energy reaches the value C, the

location

$$\{X(t,C),Y(t,C),\dot{X}(t,C),\dot{Y}(t,C): 0 \le t \le T\}$$

defined by the analytic expansions (8) and (9) in the phase space. Continuing analytically the natural family $O(C)$ means following this continuous deformation of the periodic orbit $O(C_0)$ *taken as a whole.*

In the course of the deformation, each point of the initial orbit $O(C_0)$ follows a certain path on the torus $O(C)$. It is defined in the phase space by the numerical series (7). Once this path is known, the periodic orbits can be determined unambiguously. The methods of numerical continuation propose to follow the displacement on the torus of only one point of the generating orbit. Thus they can be regarded as procedures by which the initial conditions and period of the periodic orbit $O(C_0)$ are corrected to yield the initial conditions and period for the neighboring orbits in the family.
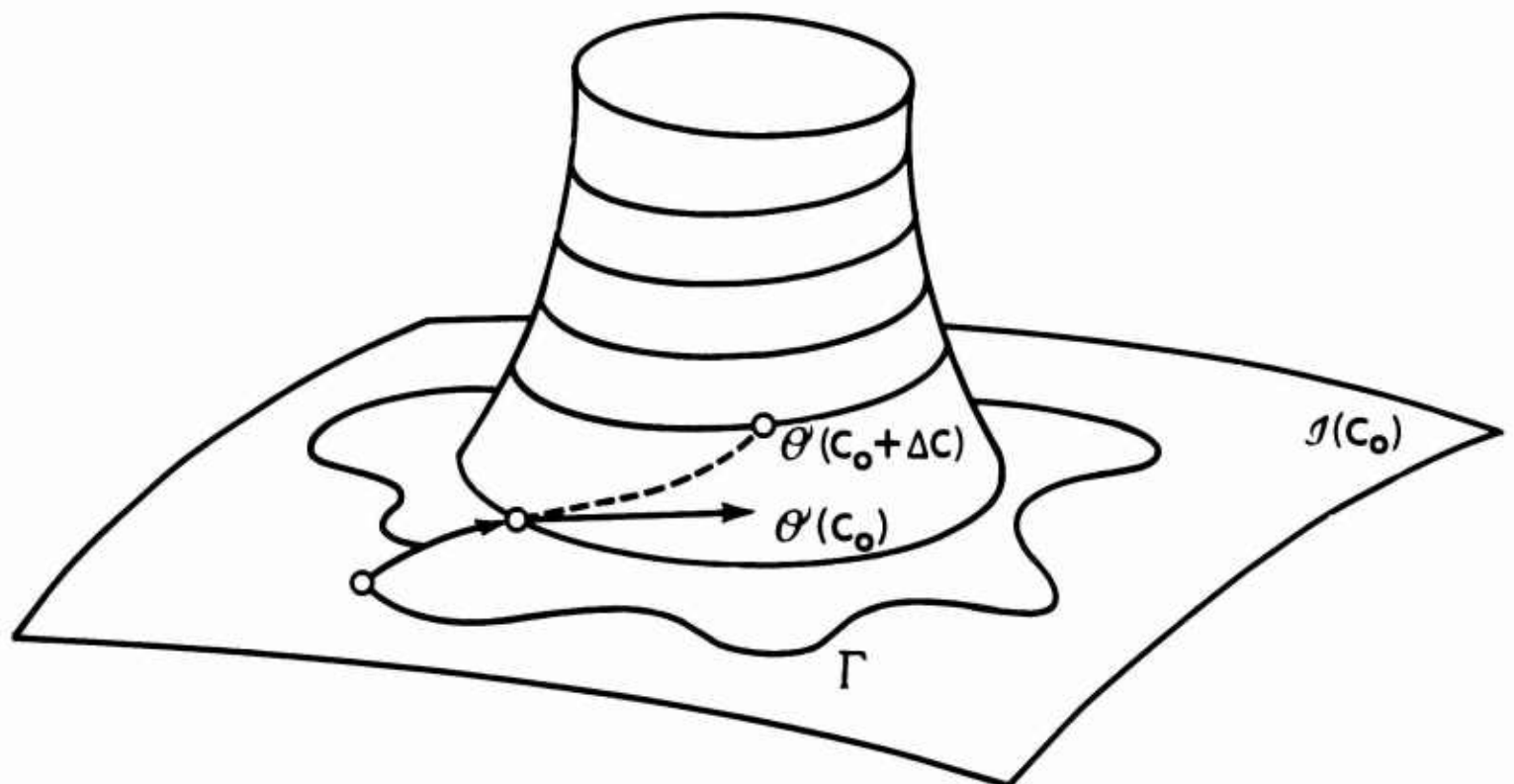

Figure 1. The two steps of a numerical continuation.

As for the methods of analytical continuation, in general it is not possible to compute all the numerical coefficients in the series (7) and (8). The methods of numerical continuation aim rather at determining, or even less at estimating approximately, only their first few coefficients, hereby determining only approximately the deformation path of the initial conditions on the torus $O(C)$. For this reason they provide basically two schemes, one for *predicting* initial conditions and a period ahead of a known periodic orbit, and the second for *correcting* the extrapolation. In regard to a natural family, both predictor and corrector should have specific properties.

Extrapolated values for the initial conditions $(x_1, y_1, \dot{x}_1, \dot{y}_1)$ and the period T_1 at the energy level $C_0 + \Delta C$ should be at least such that, to the first order,

$$x_1 = x_0 + \Delta C \cdot \frac{\partial}{\partial C} x_0(C_0),$$

$$y_1 = y_0 + \Delta C \cdot \frac{\partial}{\partial C} y_0(C_0),$$

$$\dot{x}_1 = \dot{x}_0 + \Delta C \cdot \frac{\partial}{\partial C} \dot{x}_0(C_0),$$

$$\dot{y}_1 = \dot{y}_0 + \Delta C \cdot \frac{\partial}{\partial C} \dot{y}_0(C_0),$$

$$T_1 = T_0 + \Delta C \cdot \frac{d}{dC} T_0(C_0).$$

It means that the predictor should request as a minimum a tangent to the natural family $O(C)$ at the initial position on the original orbit $O(C_0)$ in the direction of increasing energy constants, as well as the tangent at the point C_0 to the period curve $T(C)$ belonging to the family.

Evidently we could compromise and replace the tangents by the secants going through two previously evaluated sets of initial conditions on the torus. But what is gained from simplifying the predictor in this way often results in poor convergence during the corrector part of the scheme. Moreover, in approaching an element of the family which is a branching orbit on the analytic manifold $O(C)$ or an extremum on the inverse $C(T)$ of the period curve, extrapolation along the secants is apt to derail without notice the representative initial conditions from their deformation course on the family $O(C)$ onto another natural family in the neighborhood. Besides as we shall show, there is no essential difficulty in extracting the tangential elements needed for an efficient predictor from a numerical integration of the variational equations.

The phase state obtained by shifting the initial conditions along a tangent to the torus $O(C)$ by an amount equal to ΔC does not generally lie on the natural family. At any rate, it lies on the integral manifold $\mathscr{J}(C_0 + \Delta C)$ defined by the energy integral (2), and if ΔC is small enough, it lies close to the periodic orbit which is the intersection of the torus $O(C)$ and the integral manifold $\mathscr{J}(C_0 + \Delta C)$. Therefore, the corrector stage of a numerical continuation should aim at bringing this estimate of the initial conditions closer to the periodic orbit, without causing it to leave the integral manifold $\mathscr{J}$ to which it should belong. On the other hand, when only first order corrections are retained, several iterations might be necessary to correct the predicted values; therefore they should be made to converge in quadratic convergence. At last, a first order displacement of the initial conditions tangentially to the orbit (Γ) that they define is not generally going to bring the phase state closer

to the periodic orbit $O(C_0+\Delta C)$ since it amounts only to an advance
of the time on the orbit Γ. Therefore, one should use exclusively
corrections by which the phase state is displaced normally to the orbit
Γ to which it belongs. We show here how corrections having these three
properties (of being isoenergetic, quadratically convergent and normal)
can be computed by integrating numerically the homogeneous Hill's equation
for isoenergetic normal displacements as well as the accompanying
quadrature for isoenergetic tangential displacements.

2. INTRINSIC DISPLACEMENTS OF AN ORBIT

Let

$$x = x(t), \qquad y = y(t) \tag{10}$$

be a given solution of (1) defined over the time interval I. It is
assumed to have the following properties:

   a) It is not an equilibrium position;

   b) The trace of the orbit over its curve of zero velocity is
       empty.

Under these assumptions, the norm of the velocity along the orbit

$$V = (\dot{x}^2+\dot{y}^2)^{\frac{1}{2}} \tag{11}$$

is nowhere zero for as long as t is in the interval I, and the
inclination ϕ of the velocity vector on the x-axis is defined
unambiguously by the formulae

$$\dot{x} = V \cos \phi, \qquad \dot{y} = V \sin \phi. \tag{12}$$

A displacement (or variation) of the solution (10) is a pair of numerical functions

$$u \equiv u(t), \qquad v \equiv v(t)$$

that is a solution of the linear differential system

$$\dot{u} = 2A\dot{v} + (W_{xx}+2A_x\dot{y})u + (W_{xy}+2A_y\dot{y})v,$$

$$\dot{v} = -2A\dot{u} + (W_{xy}-2A_x\dot{x})u + (W_{yy}-2A_y\dot{x})v; \tag{13}$$

it is understood that the function A, its partial derivatives as well as those of W are to be expressed in (13) by means of (10) as functions of the time along the orbit.

The variational system (13) admits the integral

$$\gamma = W_x u + W_y v - \dot{x}\dot{u} - \dot{y}\dot{v}. \tag{14}$$

The normal and tangential components (n,p) of the displacement (u,v) are defined respectively by the identities

$$n = -u \sin \phi + v \cos \phi,$$

$$p = u \cos \phi + v \sin \phi. \tag{15}$$

THEOREM. *A numerical function* n = n(t) *is a normal displacement of (10) belonging to the integration constant* γ *in (14) if and only if it satisfies the linear differential equation*

$$\ddot{n} + \Theta n = 2\gamma(A + \dot{\phi})/V \tag{16}$$

where

$$\theta = \ddot{V}/V + 2(A+\dot{\phi})^2 + 2A^2 - W_{xx} - W_{yy} - 2V(A_x \sin \phi - A_y \cos \phi). \tag{17}$$

In which case, a numerical function $p = p(t)$ *is the corresponding tangential displacement of (10) for the same integration constant* γ *if and only if it is given by the quadrature*

$$\frac{d}{dt} (p/V) = 2(A+\dot{\phi})n/V - \gamma/V^2. \tag{18}$$

In order to prove the theorem, we begin by differentiating (1) with respect to the time, so as to obtain that

$$\ddot{x}v - \ddot{y}u + \dot{x}\dot{v} - \dot{y}\ddot{u} = 2A(\ddot{x}u+\ddot{y}v-\dot{x}\dot{u}-\dot{y}\dot{v})$$
$$+ (\dot{x}v-\dot{y}u)(W_{xx}+W_{yy}+2A_x\dot{y}-2A_y\dot{x}). \tag{19}$$

We now propose to reduce (19) to the equation (16). To this effect we invert the linear transformation (15) to obtain that

$$u = p \cos \phi - n \sin \phi,$$
$$v = p \sin \phi + n \cos \phi. \tag{20}$$

Then we use (12) and (20) to compute that

$$\dot{x}v - \dot{y}u = Vn; \tag{21}$$

we differentiate (12) with respect to the time and we use (20) to obtain

$$\ddot{x}u + \ddot{y}v = \dot{V}p + Vn\dot{\phi}; \tag{22}$$

also we differentiate (20) with respect to the time and calculate that

$$\dot{x}\dot{u} + \dot{y}\dot{v} = V\dot{p} - Vn\dot{\phi}. \tag{23}$$

Next we differentiate (12) twice with respect to the time so as to produce

$$\ddot{x}v - \ddot{y}u = (\ddot{V}-V\dot{\phi}^2)n - (2\dot{V}\dot{\phi}+V\ddot{\phi})p; \tag{24}$$

also we differentiate (20) twice with respect to the time, which yields

$$\ddot{u} = \ddot{p} \cos \phi - \ddot{n} \sin \phi + u\dot{\phi}^2 - 2\dot{v}\dot{\phi} - v\ddot{\phi},$$

$$\ddot{v} = \ddot{p} \sin \phi + \ddot{n} \cos \phi + v\dot{\phi}^2 + 2\dot{u}\dot{\phi} + u\ddot{\phi},$$

and thus we are able to compute that

$$\dot{x}\ddot{v} - \dot{y}\ddot{u} = V(\ddot{n}-n\dot{\phi}^2+2\dot{p}\dot{\phi}+p\ddot{\phi}). \tag{25}$$

Finally by substituting (21), (22), (23), (24) and (25) in (19), we obtain that

$$\ddot{n} + [\ddot{V}/V-2\dot{\phi}(2A+\dot{\phi})-W_{xx}-W_{yy}-2A_x\dot{y}+2A_y\dot{x}]n =$$

$$2(A+\dot{\phi})(\dot{V}p-V\dot{p})/V. \tag{26}$$

There remains now to eliminate p and $\dot{p}$ from (26). On using (1), (15) and (22), we check that

$$W_x u + W_y v = \dot{V}p + Vn\dot{\phi} + 2VAn. \tag{27}$$

Hence (23) and (27) helps writing the integral (14) in the form

$$V\dot{p} - \dot{V}p = -\gamma + 2V(A+\dot{\phi})n. \tag{28}$$

Eventually we substitute (28) in (26), and this yields the equation (16) as announced in the theorem. Of course, (28) is nothing but another form of the quadrature (18).

That, from any particular solution of (16) and any resulting quadrature (18), one is able to construct a solution of the displacement equations (13) satisfying the first integral (14) is an elementary point whose proof we leave to the reader.

Concerning the above theorem, the following comments might be appropriate:

1) Jacobi's equations (13) which determine the displacements (u,v) constitute a Lagrangian system. In that respect, the theorem can be interpreted as providing a method by which the *non-conservative* integral (14) is used in order to reduce the Lagrangian system of Jacobi's equation (originally of order 4) to a linear differential system of order 2, to be followed by a quadrature. Straightforward differentiations and eliminations were sufficient here to serve this purpose, mainly for the reason that the equations to be reduced as well as the reducing integral are linear.

2) The function Θ defined by formula (17) is well known in various problems of Celestial Mechanics.

For instance, when, in our expresssion, we take A as a constant function, then $A_x \equiv 0$ and $A_y \equiv 0$, so that

$$\Theta = \ddot{V}/V + 2(A+\dot{\phi})^2 + 2A^2 - W_{xx} - W_{yy}.$$

This expression can be found in Plummer (1911).

Also on using the differential equations (1) to compute that

$$W_x^2 + W_y^2 = \dot{V}^2 + V^2(2A+\dot{\phi})^2,$$

we deduce that

$$\ddot{x}^2 + \ddot{y}^2 = \dot{V}^2(1+\dot{\phi}^2).$$

Then, since

$$V\dddot{V} + \dot{V}^2 = \dot{x}\dddot{x} + \dot{y}\dddot{y} + \ddot{x}^2 + \ddot{y}^2,$$

we arrive at the identity

$$\frac{\dddot{V}}{V} = \dot{\phi}^2 + \frac{\ddot{x}\dot{x}+\dot{y}\ddot{y}}{V^2}.$$

Therefrom we obtain by substitution in (17) another form for Θ, namely

$$\Theta = (\ddot{x}\dddot{x}+\dot{y}\dddot{y})/V^2 + 3\dot{\phi}^2 + 4A\dot{\phi} + 4A^2 - W_{xx} - W_{yy} - 2V(A_x \sin\phi + A_y \cos\phi). \qquad (29)$$

When $A \equiv 0$, it restitutes the expression found by Poincaré (1899) while, for A being a constant function, it is the form originally given by Hill for his Lunar Theory and by Message (1959) for the planar Restricted Problem of Three Bodies.

Again on observing that

$$W_x^2 = W_y^2 + 2A(W_x\dot{y}-\dot{x}W_y) = \dot{V}^2 + V^2\dot{\phi}^2 + 2VA\dot{\phi},$$

we conclude that Θ can also be defined by the formula

$$V^2\Theta = V\ddot{V} - 2\dot{V}^2 + 2(W_x^2+W_y^2) + 4A(W_x\dot{y}-W_y\dot{x}) + 4A^2 \qquad (30)$$

$$-W_{xx} - W_{yy} - 2(A_x\dot{y}-A_y\dot{x}).$$

the problems they are concerned with (namely the mean motion of the
lunar perigree or the stability of a periodic orbit) refer exclusively
to the case when Θ is a periodic function of the time, and the only
quantities they are interested in are the characteristic exponents for
the monodromy matrix, which is the matrizant of precisely the homogeneous
equation (31) at the end of the period. Poincaré and Birkhoff make an
exception in that respect, as both adopted the Maupertuisian form of the
Principle of Least Action as the premiss from which one ought to derive
Hill's homogeneous equation (31). Thereby they were necessarily confined
to isoenergetic displacements, and from our theorem, it results quite
clearly that the Principle of Least Action can lead only to Hill's
homogeneous equation. As for the nonhomogeneous equation (16), it can
only be derived from Hamilton's general Principle of Variation wherein the
varied motion is not subject to the restriction that on it C remains
constant.

In Poincaré's derivation of Hill's homogeneous equation, several
unsatisfactory points were amended by Wintner (1930). By this expedient,
Wintner came to recognize that a direct approach could be substituted to
Poincaré's oblique treatment. However, because he still adheres to an
algebraic identity established by Poincaré, Wintner (1931a) makes his
construction unnecessarily clumsy. We like to acknowledge that our
theorem is the offshoot of an effort aiming at simplifying Wintner's
argumentation.

A straightforward deduction of the homogeneous equation (31) has
been proposed by Darwin (1897), using the arc length along the orbit as

A similar evaluation of Θ by Poincaré in the case when A is not uniformly zero was proved to be wrong by Wintner (1930). On correcting Poincaré on this point, Wintner produced the formula (30) first for the constant function A, and then generalized it to any function A which is sufficiently differentiable (Wintner 1931a).

3. ISOENERGETIC DISPLACEMENTS

Since it is linear, equation (16) has to be solved in two steps.

In the first step, one has to find the matrizant of the homogeneous equation

$$\ddot{n} + \Theta n = 0 \tag{31}$$

obtained from (16) by omitting the right-hand member. In point of fact this amounts also to putting $\gamma = 0$ in the right-hand member of (16). Therefore, the solutions of (31) with the functions determined from the resulting quadrature (18) are the normal and tangential components of displacements which belong to the particular value $\gamma = 0$ in the integral. For this reason, they are called *isoenergetic* by Wintner (1931a). Conversely, of course, the normal component of an isoenergetic displacement is a solution of the homogeneous equation (31).

This exceptional character of the isoenergetic normal displacements has not been recognized quite distinctly by Hill in his Lunar Theory. As a consequence later authors lost it out of sight, and several of them resorted to unconvincing arguments in order to justify the selection of isoenergetic displacements. Actually, in relation to the equation (16),

the independent variable. Darwin's treatment has been simplified by Plummer (1918). Professor Danby has informed us that he has extended Plummer's argumentation to encompass the nonhomogeneous equation (16).

As we know, the general solution of (31) is a linear combination

$$n(t) = \alpha n^{I}(t) + \beta n^{II}(t) \tag{32}$$

of two particular solutions satisfying respectively the initial conditions

$$n^{I}(0) = 1, \qquad \dot{n}^{I}(0) = 0, \tag{33}$$

$$n^{II}(0) = 0, \qquad \dot{n}^{II}(0) = 1. \tag{34}$$

The real numbers α and β are the two arbitary constants of integration. Besides, since the Wronskian of any pair of solutions for (31) is a constant, we have that

$$n^{I}(t)\dot{n}^{II}(t) - \dot{n}^{I}(t)n^{II}(t) = 1. \tag{35}$$

Therefore, the matrizant of (31) which is the matrix

$$N(t) = \begin{pmatrix} n^{I}(t) & n^{II}(t) \\ \dot{n}^{I}(t) & \dot{n}^{II}(t) \end{pmatrix} \tag{36}$$

is unimodular, so that its inverse is simply the matrix

$$N^{-1}(t) = \begin{pmatrix} \dot{n}^{II}(t) & -n^{II}(t) \\ -\dot{n}^{I}(t) & n^{I}(t) \end{pmatrix} .$$

Consequently from the expression of the general solution of (16), namely

$$\begin{pmatrix} n(t) \\ \dot{n}(t) \end{pmatrix} = N(t) \begin{pmatrix} n_0 \\ \dot{n}_0 \end{pmatrix} + 2\gamma N(t) \int_0^t N^{-1}(s) \begin{pmatrix} 0 \\ [A(s)+\dot{\phi}(s)]/V(s) \end{pmatrix} ds,$$

we conclude that the function

$$n(t) = n^I(t)n_0 + n^{II}(t)\dot{n}_0 + 2\gamma[n^I(t)a(t)+n^{II}(t)b(t)] \tag{37}$$

where

$$a(t) = - \int_0^t n^{II}(s)[A(s)+\dot{\phi}(s)]/V(s)ds$$

$$\tag{38}$$

$$b(t) = \int_0^t n^I(s)[A(s)+\dot{\phi}(s)]/V(s)ds$$

is, among all normal displacements defined by the initial conditions n_0 and $\dot{n}_0$, the one which gives the value γ to the integral (14).

In most instances, neither Θ nor the right-hand member of (16) can be produced in literal form, hence the basic isoenergetic normal displacements n^I and n^{II} can be obtained only by numerical integration. As a result, one should look for a better way of obtaining the general normal displacement (32) than by performing the quadratures (38). Let us consider the equation

$$\ddot{n} + \Theta n = 2(A+\dot{\phi})/V \tag{39}$$

and denote by n^{III} its solution determined by the initial conditions

$$n^{III}(0) = \dot{n}^{III}(0) = 0.$$

Then γn^{III} is the particular solution of (31) which satisfies the initial conditions $n(0) = \dot{n}(0) = 0$, and, in view of (37), the general solution of (16) can also be given the form

$$n = \alpha n^{I} + \beta n^{II} + \gamma n^{III} \tag{40}$$

where α and β are two arbitrary constants of integration.

4. ISOENERGETIC CORRECTOR

Let us assume that x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ are approximately the initial conditions for a periodic orbit belonging to the energy constant C_0; let T denote an approximate estimate of its period. We represent by $x(t)$, $y(t)$ the solution of the equations (1) which satisfies the initial conditions x_0, y_0, $\dot{x}_0$ and $\dot{y}_0$.

The problem is to determine the first order corrections $u(t)$, $v(t)$ such that

$$X(t) = x(t) + u(t), \qquad Y(t) = y(t) + v(t)$$

be, for the same Painlevé constant C_0, a closer approximation to the periodic orbit. The corrections to the initial orbit imply that the approximate period T will be modified at the first order by an amount ΔT.

Let ϕ_0 denote the inclination on the x-axis of the velocity at the initial time $t = 0$ along the initial orbit. By orthogonal projections on the initial tangent and the initial normal, we obtain the periodicity conditions in the following form:

$$X(T+\Delta T)\cos\,\phi_0 + Y(T+\Delta T)\sin\,\phi_0 = X(0)\cos\,\phi_0 + Y(0)\sin\,\phi_0,$$

$$-X(T+\Delta T)\sin\,\phi_0 + Y(T+\Delta T)\cos\,\phi_0 = -X(0)\sin\,\phi_0 + Y(0)\cos\,\phi_0,$$

$$\dot{X}(T+\Delta T)\cos\,\phi_0 + \dot{Y}(T+\Delta T)\sin\,\phi_0 = \dot{X}(0)\cos\,\phi_0 + \dot{Y}(0)\sin\,\phi_0,$$

$$-\dot{X}(T+\Delta T)\sin\,\phi_0 + \dot{Y}(T+\Delta T)\cos\,\phi_0 = -\dot{X}(0)\sin\,\phi_0 + \dot{Y}(0)\cos\,\phi_0.$$

In these expressions, we perform the substitutions

$$X(T+\Delta T) = x(T) + \dot{x}(T)\Delta T + u(T),$$

$$Y(T+\Delta T) = y(T) + \dot{y}(T)\Delta T + v(T),$$

$$\dot{X}(T+\Delta T) = \dot{x}(T) + \ddot{x}(T)\Delta T + \dot{u}(T),$$

$$\dot{Y}(T+\Delta T) = \dot{y}(T) + \ddot{y}(T)\Delta T + \dot{v}(T).$$

On omitting terms of order higher than one, we arrive at the correction equations

$$[u(T)-u(0)]\cos\,\phi_0 + [v(T)-v(0)]\sin\,\phi_0 + V_0\Delta T = \Delta_1,$$

$$[u(T)-u(0)]\sin\,\phi_0 - [v(T)-v(0)]\cos\,\phi_0 = \Delta_2,$$

$$[\dot{u}(T)-\dot{u}(0)]\cos\,\phi_0 + [\dot{v}(T)-\dot{v}(0)]\sin\,\phi_0 + \dot{V}_0\Delta T = \Delta_3, \qquad (41)$$

$$[\dot{u}(T)-\dot{u}(0)]\sin\,\phi_0 - [\dot{v}(T)-\dot{v}(0)]\cos\,\phi_0 - V_0\dot{\phi}_0\Delta T = \Delta_4,$$

where we put

$$\Delta_1 = -[x(T)-x_0]\cos\,\phi_0 - [y(T)-y_0]\sin\,\phi_0,$$

$$\Delta_2 = -[x(T)-x_0]\sin\,\phi_0 + [y(T)-y_0]\cos\,\phi_0,$$

$$\Delta_3 = -[\dot{x}(T)-\dot{x}_0]\cos\,\phi_0 - [\dot{y}(T)-\dot{y}_0]\sin\,\phi_0, \qquad (42)$$

$$\Delta_4 = -[\dot{x}(T)-\dot{x}_0]\sin\,\phi_0 + [\dot{y}(T)-\dot{y}_0]\cos\,\phi_0.$$

The corrections will be decomposed into their normal and tangential components along the initial orbit so that, at the first order, we have that

$$u(T) - u(0) = [p(T)-p(0)]\cos \phi_0 - [n(T)-n(0)]\sin \phi_0,$$

$$v(T) - v(0) = [p(T)-p(0)]\sin \phi_0 + [n(T)-n(0)]\cos \phi_0,$$

$$\dot{u}(T) - \dot{u}(0) = [\dot{p}(T)-\dot{p}(0)]\cos \phi_0 - [\dot{n}(T)-\dot{n}(0)]\sin \phi_0$$

$$- \dot{\phi}_0\{[p(T)-p(0)]\sin \phi_0 + [n(T)-n(0)]\cos \phi_0\}, \qquad (43)$$

$$\dot{v}(T) - \dot{v}(0) = [\dot{p}(T)-\dot{p}(0)]\sin \phi_0 + [\dot{n}(T)-\dot{n}(0)]\cos \phi_0$$

$$+ \dot{\phi}_0\{[p(T)-p(0)]\cos \phi_0 - [n(T)-n(0)]\sin \phi_0\}.$$

We substitute the expressions (43) in the correction equations (41) and, after a few manipulations, we arrive at the system

$$n(T) - n(0) = -\Delta_2,$$
$$\dot{n}(T) - \dot{n}(0) = -\Delta_4 - \Delta_1\dot{\phi}_0,$$
$$p(T) - p(0) + V_0\Delta T = \Delta_1, \qquad (44)$$
$$\dot{p}(T) - \dot{p}(0) + \dot{V}_0\Delta T = \Delta_3 - \Delta_2\dot{\phi}_0.$$

We should now observe that these four correction equations are not independent. Indeed from the variational integral in the form (28), we deduce immediately that

$$V_0[\dot{p}(T)-\dot{p}(0)] = \dot{V}_0[p(T)-p(0)] + 2V_0(A_0+\dot{\phi}_0)[n(T)-n(0)],$$

which proves that, at the first order, the fourth of the equations (44) ought to be a linear combination of the first and the third one. This implies that, at the first order, we ought to have

$$\dot{V}_0 \Delta_1 - V_0 (2A_0 + \dot{\phi}_0) \Delta_2 - V_0 \Delta_3 = 0. \tag{45}$$

Thus by computing this quantity at the period T along the initial orbit, we shall gain an estimate of the second order contributions to the corrections, which contributions we have decided to omit from the corrector scheme. And it is one of the aims of the iteration on the corrections to bring these contributions down as close to zero as possible.

The variational integral (28) informs us also that for the corrections on the initial orbit to be isoenergetic to the first order, it is necessary and sufficient that the displacements n and p be in turn isoenergetic. As we settled down for an isoenergetic corrector, from now on in this paragraph, we shall restrict ourselves to isoenergetic displacements.

Thus the normal correction will present itself as the linear combination

$$n = \alpha n^I + \beta n^{II}. \tag{46}$$

Consequently, since the quadrature (18) is linear in p and n, the tangential correction will be of the form

$$p = \alpha p^I + \beta p^{II} \tag{47}$$

where p^I (resp. p^{II}) results from n^I (resp. n^{II}) in virtue of the quadrature

$$\frac{d}{dt} \left(\frac{p}{V} \right) = 2(A + \dot{\phi}) \frac{n}{V} . \tag{48}$$

Before we fix the initial conditions $p^I(0)$ and $p^{II}(0)$ requested by the quadratures, we should realize that, since the dynamical system is conservative, the selection of an initial value $p(0)$ for the tangential displacement amounts to nothing else than a translation of the time origin, which is an irrelevant operation when the problem is to determine the initial conditions and the period of a periodic orbit. Therefore, without any loss of consistency, we can impose that

$$p^I(0) = p^{II}(0) = 0. \tag{49}$$

In geometrical terms, the only displacement that our corrector provides for the position is a shift along the normal to the original orbit at the initial position.

On using the initial conditions (33), (34), (49) and on substituting (46) and (47) in the correction equations (44), we readily obtain that the corrective displacement α along the normal and the modification β for the initial normal velocity are solutions of the linear system

$$\alpha[n^I(T)-1] + \beta n^{II}(T) = -\Delta_2,$$
$$\alpha\dot{n}^I(T) + \beta[\dot{n}^{II}(T)-1] = -\Delta_4 - \Delta_1\dot{\phi}_0. \tag{50}$$

In view of (35), its determinant is found to be equal to

$$2 - [n^I(T)+\dot{n}^{II}(T)].$$

Thus the cases when the trace

$$\mathrm{Tr}(T) = n^I(T) + \dot{n}^{II}(T)$$

of the matrizant $N(T)$ for Hill's homogeneous equation is equal to 2,
turn out to be singularities for our corrector scheme. As we rule them
out of order here, we may assume that the system (50) has a unique
solution. Then much in the same way as we did for the system (50), we
extract from the third of the equations (44) the first order correction
to the period:

$$\Delta T = [\Delta_1 - \alpha p^I(T) - \beta p^{II}(T)]/V_0. \tag{51}$$

Once the factors α and β have been determined, we can compute
the corrections to the initial conditions in Cartesian coordinates through
the following sequence of formulae

$$\Delta x_0 = u(0) = -\alpha \sin \phi_0,$$
$$\Delta y_0 = v(0) = \alpha \cos \phi_0,$$
$$\Delta \dot{x}_0 = \dot{u}(0) = \alpha[2A(x_0,y_0)+\dot{\phi}_0]\cos \phi_0 - \beta \sin \phi_0,$$
$$\Delta \dot{y}_0 = \dot{v}(0) = \alpha[2A(x_0,y_0)+\dot{\phi}_0]\sin \phi_0 + \beta \cos \phi_0.$$

On inserting the new initial positions

$$x_0 + \Delta x_0, \quad y_0 + \Delta y_0$$

into Painlevé's integral (2) where the constant C is given the value
C_0, we compute the new norm

$$V_0^* = [2W(x_0+\Delta x_0,y_0+\Delta y_0)-C_0]^{\frac{1}{2}}$$

of the velocity vector. But the quantities

$$\dot{x}_0 + \Delta \dot{x}_0, \quad \dot{y}_0 + \Delta \dot{y}_0$$

give the new direction of this vector. In consequence, we compute the norm of this direction vector

$$\bar{V}_0 = [(\dot{x}_0 + \Delta\dot{x}_0)^2 + (\dot{y}_0 + \Delta\dot{y}_0)^2]^{\frac{1}{2}}$$

so that for the new initial velocities we obtain the numbers

$$V_0^* \frac{\dot{x}_0 + \Delta\dot{x}_0}{\bar{V}_0} , \qquad V_0^* \frac{\dot{y}_0 + \Delta\dot{y}_0}{\bar{V}_0} .$$

5. TANGENTIAL PREDICTOR

Given a periodic orbit

$$x(t;t_0,x_0,y_0,\dot{x}_0,\dot{y}_0;C_0), \qquad y(t;t_0,x_0,y_0,\dot{x}_0,\dot{y}_0;C_0)$$

with period T_0, which is determined at time t_0 by the initial conditions $(x_0,y_0,\dot{x}_0,\dot{y}_0)$ such that Painlevé's constant takes the value C_0, let us assume that, for this orbit, the matrix (6) is of rank 4. Thus there exists in the neighborhood of C_0 a natural family $O(C)$ of periodic orbits which contains the given periodic orbit. Moreover, since the continuation $O(C)$ is analytic, the family is represented by expansions of the form

$$X(t;C_0+\Delta C) = x(t;C_0) + \Delta C \cdot \frac{\partial}{\partial C} x(t;C_0) + \cdots,$$

$$Y(t;C_0+\Delta C) = y(t;C_0) + \Delta C \cdot \frac{\partial}{\partial C} y(t;C_0) + \cdots,$$

(53)

and the period T along the natural family likewise takes the form of a series in ΔC:

$$T(C_0+\Delta C) = T_0 + \Delta C \cdot \frac{d}{dC} T_0 + \cdots . \tag{54}$$

The first order of the series (53) have to check to the first order the variational integral

$$W_x(x(t;C_0),y(t;C_0)) \frac{\partial}{\partial C} x(t;C_0) + W_y(x(t;C_0),y(t;C_0)) \frac{\partial}{\partial C} y(t;C_0)$$

$$- \dot{x}(t;C_0) \frac{\partial}{\partial C} \dot{x}(t;C_0) - \dot{y}(t;C_0) \frac{\partial}{\partial C} y(t;C_0) = \tfrac{1}{2}.$$

Basically we do concern ourselves here with the problem of actually computing as function of the time the displacement $(\partial x(t;C_0)/\partial C, \partial y(t;C_0)/\partial C)$ which causes the varied orbit to be periodic with the modified period (54). For the sake of convenience, we rather put

$$\gamma = \tfrac{1}{2}\Delta C, \quad u = 2 \frac{\partial x}{\partial C}, \quad v = 2 \frac{\partial y}{\partial C}, \quad \Delta T = 2 \frac{dT_0}{\partial C} .$$

In these notations, the problem can be reformulated as that of finding a displacement (u,v) which satisfies the variational integral

$$W_x u + W_y v - \dot{x}\dot{u} - \dot{y}\dot{v} = 1 \tag{55}$$

and is such that, for γ sufficiently small, the orbit

$$X(t) = x(t) + \gamma u(t) + \cdots \qquad Y(t) = y(t) + \gamma v(t) + \cdots$$

be periodic with period

$$T = T_0 + \gamma \Delta T.$$

Exactly as we did when we set up the corrector scheme, we project the first order conditions of periodicity orthogonally onto the tangent and the normal at the initial point of the given orbit $O(C_0)$. Introducing

therein the tangential and normal components of the displacement, we eventually arrive at the following conditions of periodicity

$$n(T) - n(0) = 0,$$
$$\dot{n}(T) - \dot{n}(0) = 0,$$
$$p(T) - p(0) + V_0\Delta T = 0,$$
$$\dot{p}(T) - \dot{p}(0) + \dot{V}_0\Delta T = 0.$$

(56)

But, in view of (55), the normal and tangential displacements satisfy the first integral

$$V\dot{p} - \dot{V}p - 2V(A+\dot{\phi})n = 1.$$

(57)

It implies in particular that

$$V_0[\dot{p}(T)-\dot{p}(0)] = \dot{V}_0[p(T)-p(0)] + 2V_0(A_0+\dot{\phi}_0)[n(T)-n(0)],$$

so that the third condition of periodicity turns out to be a linear combination of the first and the second one.

Now from (57), we also conclude that the normal displacement is a solution of Hill's nonhomogeneous equation

$$\ddot{n} + \Theta n = 2(A+\phi)/V.$$

(58)

Then, as we have shown above, it is a linear combination of the type

$$n = \alpha n^{I} + \beta n^{II} + n^{III},$$

where n^{I} and n^{II} are the basic solution of Hill's homogeneous equation, and n^{III} is the particular solution of (58) for the initial

conditions $n^{III}(0) = \dot{n}^{III}(0) = 0$. Correspondingly the tangential displacement is the linear combination

$$p = \alpha p^{I} + \beta p^{II} + p^{III}$$

where p^{III} is given by the quadrature

$$\frac{d}{dt}\left(\frac{p^{III}}{V}\right) = 2\,\frac{A+\dot{\phi}}{V}\,n^{III} - \frac{1}{V^2}$$

for the usual initial conditon $p^{III}(0) = 0$.

As a result of all preceding remarks, the conditions of periodicity (56) becomes

$$\begin{aligned}
\alpha[n^{I}(T)-1] + \beta n^{II}(T) &= -n^{III}(T), \\
\alpha\dot{n}^{I}(T) + \beta[\dot{n}^{II}(T)-1] &= -\dot{n}^{III}(T), \\
\alpha p^{I}(T) + \beta p^{II}(T) + V_0\Delta T &= -p^{III}(T).
\end{aligned} \qquad (59)$$

The determinant of the first two equations is equal to $2 - \mathrm{Tr}(T)$. For the sake of brevity we shall not discuss here the singular case of a periodic orbit for which $\mathrm{Tr}(T) = 2$, i.e. of an orbit whose characteristic exponent is equal to zero. Let it be mentioned that it occurs only in exceptional circumstances, like a relative extremum for the period function $T(C)$, a bifurcation on the natural family $O(C)$ or even an essential singularity beyond which no analytical continuation of the manifold is permissible, which is what Wintner (1931a) calls a *natural termination* of the family. In principle these situations have been analyzed by Poincaré (1899). Recently they have received more careful attention from Hénon (1965) in relation to several natural families of synodically symmetric orbits in the Restricted Problem of Three Bodies.

Once the coefficients α and β have been obtained, we can compute the derivations at $C = C_0$ of the initial conditions which determine the periodic orbit $O(C_0)$, namely

$$\frac{d}{dc} x_0(C_0) = \frac{1}{2} u(0) = -\frac{1}{2} \alpha \sin \phi_0,$$

$$\frac{d}{dc} y_0(C_0) = \frac{1}{2} v(0) = \frac{1}{2} \alpha \cos \phi_0,$$

$$\frac{d}{dC} \dot{x}_0(C_0) = \frac{1}{2} \dot{u}(0) = -\frac{1}{2} \frac{\cos \phi_0}{V_0} + \frac{1}{2}(2A_0 + \dot{\phi}_0)\alpha \cos \phi_0 - \frac{1}{2} \beta \sin \phi_0,$$

$$\frac{d}{dc} \dot{y}_0(C_0) = \frac{1}{2} \dot{v}(0) = -\frac{1}{2} \frac{\sin \phi_0}{V_0} + \frac{1}{2}(2A_0 + \dot{\phi}_0)\alpha \sin \phi_0 + \frac{1}{2} \beta \cos \phi_0.$$

In geometric terms, we hold here a tangent to the manifold $O(C)$ at the point C_0. Thus in order to obtain a first order approximation to the next orbit in the family which lies on the manifold $\mathcal{I}(C_0 + C)$ determined by the integral (2), we suggest moving the initial point to the point whose coordinates are

$$x_0^* = x_0 + \Delta C \cdot \frac{d}{dc} x_0(C_0), \qquad y_0^* = y_0 + \Delta C \cdot \frac{d}{dc} y_0(C_0).$$

Then we compute the norm of the velocity vector from Painlevé's integral:

$$V_0^* = [2W(x_0^*, y_0^*) - C_0 - \Delta C]^{\frac{1}{2}}.$$

At last we estimate the direction of this vector by the two numbers

$$\bar{x}_0 = \dot{x}_0 + \Delta C \cdot \frac{d}{dc} \dot{x}_0(C_0), \qquad \bar{y}_0 = \dot{y}_0 + \Delta C \cdot \frac{d}{dc} \dot{y}_0(C_0),$$

so that, if we put

$$\bar{v}_0 = (\bar{x}_0^2 + \bar{y}_0^2)^{\frac{1}{2}},$$

the new initial velocity will have for components

$$\dot{x}_0^* = V_0^* \frac{\bar{x}_0}{\bar{v}_0}, \qquad \dot{y}_0^* = V_0^* \frac{\bar{y}_0}{\bar{v}_0}.$$

At this stage, then we resort to the corrector scheme to move the newly obtained initial conditions (point B in Fig. 1) by successive approximations along the manifold $\mathcal{J}(C_0+\Delta C)$ right onto those of the periodic orbit $O(C_0+\Delta C)$ (point A' in Fig. 1).
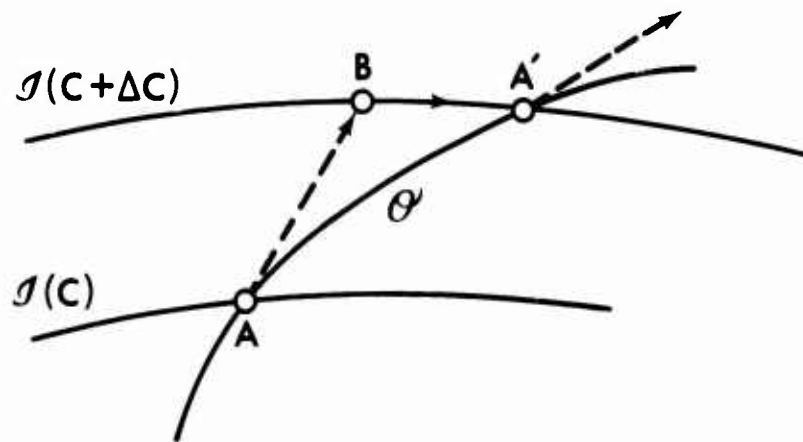

Figure 2. Diagram of numerical continuation for a natural family of periodic orbits.

6. CHARACTERISTIC EXPONENTS

As a by-product of our numerical computation, we collect immediately the nontrivial characteristic exponents of each periodic orbit that we compute.

Let ω be one of these exponents. There exists a displacement of the orbit which is of the form

$$u(t) = e^{\omega t}U(t), \qquad v(t) = e^{\omega t}V(t) \tag{61}$$

where U and V are periodic functions with period T. We prove that the displacement (61) is isoenergetic. Indeed, along the varied orbit, the first order variation of the energy integral is of the form

$$\gamma = P(t)e^{\omega t} \tag{62}$$

where P is a periodic function with period T. Since this variation is an integral of the variational equations, (13) implies that P is identically equal to zero, which means that the constant γ is equal to zero, and this establishes that (61) is an isoenergetic displacement.

Now the normal component of (61) is of the form

$$n(t) = e^{\omega t}N(t) \tag{63}$$

where N is a periodic function with period T. But (63) expresses precisely that ω is a characteristic exponent of Hill's homogeneous equation.

It follows that, in order to compute the nontrivial characteristic exponents of a periodic orbit, it is necessary and sufficient to

calculate the characteristic exponents of Hill's equation associated with it.

But as we know, these quantities are such that

$$s_1 = e^{\omega T} \quad \text{and} \quad s_2 = e^{-\omega T}$$

are the eigenvalues of the matrizant $N(T)$ given by (36) at the end of the period of the function Θ, which is precisely the period of the orbit. Therefore s_1 and s_2 are the roots of the characteristic equation

$$s^2 - [n^I(T) + \dot{n}^{II}(T)]s + [n^I(T)\dot{n}^{II}(T) - n^{II}(T)\dot{n}^I(T)] = 0.$$

In view of the Wronskian integral (35), it can even be written simply as

$$s^2 - Tr(T)s + 1 = 0. \tag{64}$$

Consequently, the trace $Tr(T)$ can be used as an *index of the stability* of the orbit.

a) If $|Tr(T)| > 2$, the characteristic exponents of the orbit are of the unstable type.

b) If $|Tr(T)| < 2$, they are of the stable type.

c) $Tr(T) = +2$ or -2 represents what Wintner terms the two cases of indifferent stability.

7. SYMMETRIC ORBITS

Let us assume that the functions W and A which characterize the dynamical system have the following symmetry property:

$$W(x,-y) = W(x,y), \quad A(x,-y) = A(x,y)$$

identically in the configuration plane.

Then the substitutions

$$t \rightarrow -t, \qquad y \rightarrow -y$$

leave the equations of motion unchanged. This invariance in turn implies that, if the pair $x(t), y(t)$ is a solution of the equations of motion (1), then the pair $\xi(t), \eta(t)$ such that

$$\xi(t) = x(-t), \qquad \eta(t) = -y(-t)$$

is also a solution ("Principle of Symmetry"). In particular, a solution defined by the initial conditions $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ such that

$$y_0 = \dot{x}_0 = 0 \tag{65}$$

is *symmetric* with respect to the x-axis. Indeed, in view of Cauchy's theorem of uniqueness, the Principle of Symmetry results here in the identities

$$x(-t) = x(t), \qquad y(-t) = -y(t). \tag{66}$$

It is easily checked that the conditions (66) which define an orbit symmetric with respect to the x-axis imply the following identities

$$V(-t) = V(t), \qquad \cos \phi(-t) = -\cos \phi(t),$$
$$\dot{\phi}(-t) = \dot{\phi}(t), \qquad \sin \phi(-t) = \sin \phi(t).$$

As a result, along such a symmetric orbit, the coefficient Θ in Hill's equation is an even function of the time.

Now we propose to show that, along the symmetric orbit defined by the initial conditions (65), the isoenergetic normal displacement n^I (*resp.* n^{II}) is an *even* function (*resp.* an *odd* function) of the time.

Indeed the resolvent $N(t;0)$ of Hill's equation is characterized as the solution, defined by the initial condition

$$N(0;0) = I_2,$$

to the differential matrix equation

$$\dot{N}(t;0) = H(t)N(t;0) \tag{67}$$

where

$$H(t) = \begin{pmatrix} 0 & 1 \\ \Theta(t) & 0 \end{pmatrix}. \tag{68}$$

Since $\Theta(-t) = \Theta(t)$, the matrix function $H(t)$ is an *even* function of the time. Therefore, the substitution $t \to -t$ in (67) provides the identity

$$\dot{N}(-t;0) = -H(t)N(-t;0). \tag{69}$$

If we put

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and observe that

$$S^2 = I_2, \qquad SH(t)S = -H(t),$$

we see that (69) can also be written in the form

$$\frac{d}{dt} [SN(-t;0)] = H(t)[SN(-t;0)].$$

Hence, on using the fact that $N(t;0)$ is the resolvent of (67), we come at last to the identity

$$SN(-t;0) = N(t;0)S. \tag{70}$$

But the matrix identity (70) is equivalent to the scalar identities

$$n^I(-t) = n^I(t), \qquad n^{II}(-t) = -n^{I\cdot}(t). \tag{71}$$

This concludes the proof of the proposition.

From (71) together with (18), it results that

$$p^I(-t) = -p^I(t), \qquad p^{II}(-t) = p^{II}(t). \tag{72}$$

Now that we have reviewed the symmetry properties of the displacements of a symmetric orbit, we impose the condition that the symmetric orbit be also periodic. Let T be its period.

Because the matrix function $H(t)$ is periodic with the same period T, on substituting $t + T$ for t in (67), we obtain that

$$\dot{N}(t+T;0) = H(t)N(t+T;0)$$

and deduce therefrom that

$$N(T+t;0) = N(t;0)N(T;0)$$

at any time t. In particular, for $t = -T/2$,

$$N(T;0) = N^{-1}(-T/2;0)N(T/2;0).$$

This matrix formula gives rise to the scalar expressions

$$n^I(T) = \dot{n}^{II}(T) = n^I(T/2)\dot{n}^{II}(T/2) + n^{II}(T/2)\dot{n}^I(T/2), \qquad (73)$$

$$\dot{n}^I(T) = 2n^I(T/2)\dot{n}^I(T/2), \qquad (74)$$

$$n^{II}(T) = 2n^{II}(T/2)\dot{n}^{II}(T/2). \qquad (75)$$

Note that the Wronskian integral taken at half the period,

$$n^I(T/2)\dot{n}^{II}(T/2) - \dot{n}^I(T/2)n^{II}(T/2) = 1,$$

gives for (73) an expression

$$n^I(T) = \dot{n}^{II}(T) = 1 - 2\dot{n}^I(T/2)^{II}(T/2) \tag{76}$$

that will be useful later on.

Formula (73) is especially interesting for the computation of the characteristic exponent of a symmetric periodic orbit. Indeed, as we have seen before,

$$2 \cosh \omega T = Tr(T) = n^I(T) + \dot{n}^{II}(T),$$

so that, in view of (73),

$$\cosh \omega T = n^I(T/2)\dot{n}^{II}(T/2) + \dot{n}^I(T/2)n^{II}(T/2). \tag{77}$$

Thus integrating the orbital equations and Hill's homogeneous equation for *only half a period* is sufficient to compute the characteristic exponent ω of a symmetric periodic orbit.

This proposition which was known to Darwin (1897) has been rediscovered by Moulton (1914) and applied extensively by Lemaître in his explorations of the symmetric periodic solutions to Störmer's problem. We plan to show that, like for the computation of the characteristic exponent, the numerical continuation of a natural family consisting of symmetric orbits requires the orbit and its intrinsic variations to be calculated over only half the period.

In order to do so, we need to back up and consider the problem of converging iteratively toward the initial conditions determining a

non-necessarily symmetric periodic orbit. Here we shall start from the periodicity conditions

$$X(T/2+\Delta T/2) = X(-T/2-\Delta T/2),$$

$$Y(T/2+\Delta T/2) = Y(-T/2-\Delta T/2),$$

$$\dot{X}(T/2+\Delta T/2) = \dot{X}(-T/2-\Delta T/2),$$

$$\dot{Y}(T/2+\Delta T/2) = \dot{Y}(-T/2-\Delta T/2).$$

To the first order in the variations, they generate the correction equations:

$$u(T/2) - u(-T/2) + [\dot{x}(T/2)+\dot{x}(-T/2)]\Delta T/2 = -[x(T/2)-x(-T/2)],$$

$$v(T/2) - v(-T/2) + [\dot{y}(T/2)+\dot{y}(-T/2)]\Delta T/2 = -[y(T/2)-y(-T/2)],$$

$$\dot{u}(T/2) - \dot{u}(-T/2) + [\ddot{x}(T/2)+\ddot{x}(-T/2)]\Delta T/2 = -[\dot{x}(T/2)-\dot{x}(-T/2)],$$

$$\dot{v}(T/2) - \dot{v}(-T/2) + [\ddot{y}(T/2)+\ddot{y}(-T/2)]\Delta T/2 = -[\dot{y}(T/2)-\dot{y}(-T/2)].$$

By orthogonal projection onto the tangent and the normal to the orbit at the time $T/2$, we obtain from them that

$$p(T/2) - p(-T/2) + V(T/2)\Delta T = \Delta_1,$$

$$n(T/2) - n(-T/2) \qquad\qquad = -\Delta_2,$$

$$\dot{p}(T/2) - \dot{p}(-T/2) + \dot{V}(T/2)\Delta T = \Delta_3 - \Delta_2\dot{\phi}(T/2),$$

$$\dot{n}(T/2) - \dot{n}(-T/2) \qquad\qquad = -\Delta_4 - \Delta_1\dot{\phi}(T/2),$$

(79)

where we have put

$$\Delta_1 = -[x(T/2)-x(-T/2)]\cos \phi(T/2) - [y(T/2)-y(-T/2)]\sin \phi(T/2),$$

$$\Delta_2 = -[x(T/2)-x(-T/2)]\sin \phi(T/2) + [y(T/2)-y(-T/2)]\cos \phi(T/2),$$

$$\Delta_3 = -[\dot{x}(T/2)-\dot{x}(-T/2)]\cos \phi(T/2) - [\dot{y}(T/2)-\dot{y}(-T/2)]\sin \phi(T/2),$$

$$\Delta_4 = -[\dot{x}(T/2)-\dot{x}(-T/2)]\sin \phi(T/2) + [\dot{y}(T/2)-\dot{y}(-T/2)]\cos \phi(T/2).$$

(80)

If we use the correction equations (79) to set up an isoenergetic corrector, we have to assume that

$$n = \alpha n^I + \beta n^{II} \quad \text{and} \quad p = \alpha p^I + \beta p^{II},$$

in which case the formulae (79) become

$$\alpha[n^I(T/2)-n^I(-T/2)] + \beta[n^{II}(T/2)-n^{II}(-T/2)] = -\Delta_2,$$

$$\alpha[\dot{n}^I(T/2)-\dot{n}^I(-T/2)] + \beta[\dot{n}^{II}(T/2)-\dot{n}^{II}(-T/2)] = -\Delta_4 - \Delta_1\dot{\phi}(T/2),$$

$$\alpha[p^I(T/2)-p^I(-T/2)] + \beta[p^{II}(T/2)-p^{II}(-T/2)] + V(T/2)\Delta T = \Delta_1,$$

$$\alpha[\dot{p}^I(T/2)-\dot{p}^I(-T/2)] + \beta[\dot{p}^{II}(T/2)-\dot{p}^{II}(-T/2)] + \dot{V}(T/2)\Delta T = \Delta_3 - \Delta_2\dot{\phi}(T/2).$$

(81)

Let us now assume that the generating orbit $x(t),y(t)$ is symmetric with respect to the axis Ox, its initial conditions satisfying the relations (65). Then, in view of the symmetry identities (66), we obtain that, to the first order in the displacements,

$$\cos \phi(T/2) = 0$$

which implies that

$$\Delta_1 = -2y(T/2)\sin \phi(T/2), \qquad \Delta_2 = 0,$$

$$\Delta_4 = -2\dot{x}(T/2)\sin \phi(T/2), \qquad \Delta_3 = 0.$$

Taking also into account the symmetry identities (71) and (72), we
are thus able to reduce the correction equations (81) to the simpler
system

$$\beta n^{II}(T/2) = 0,$$
$$\alpha \dot{n}^{I}(T/2) = [\dot{x}(T/2)+y(T/2)\dot{\phi}(T/2)]\sin\phi(T/2),$$
$$\alpha p^{I}(T/2) + \frac{1}{2}V(T/2)\Delta T = -y(T/2)\sin\phi(T/2),$$
$$\beta \dot{p}^{II}(T/2) = 0.$$

(82)

This system needs to be discussed. If $n^{II}(T/2) = 0$, then β
is left undetermined by the first equation. However, in view of (75)
and (76), we have in this case that $n^{II}(T) = 0$ and $n^{I}(T) = 1$, so
that we find ourselves here in the critical case where $Tr(T) = 2$.
Moreover, from the variational integral (28) along an isoenergetic
displacement $(\gamma = 0)$, we also have that $\dot{p}^{II}(T/2) = 0$, since, to the
first order, $\dot{V}(T/2) = 0$. Thus the fourth equation in (82) is
compatible. Conversely, if $\dot{p}^{II}(T/2) = 0$, then we can show in the same
way that $n^{II}(T/2) = 0$. Hence, if we exclude the critical case when
$Tr(T) = 2$, we must have that $\beta = 0$.

The second of the equations (82) has a unique solution if and only
if $\dot{n}^{I}(T/2) \neq 0$. But, in view of (74) and (76) $\dot{n}^{I}(T/2) = 0$ implies
that $n^{I}(T) = 0$ and consequently that $Tr(T) = 2$. Therefore, excluding
once more the critical case, we can extract from the second of the
equations (82) the correction displacement α and thereafter from the
third equation the correction on the period. It is then obvious that
the corrections on the initial conditions are

$$\Delta x_0 = -\alpha \sin \phi_0,$$

$$\Delta y_0 = 0,$$

$$\Delta \dot{x}_0 = 0,$$ (83)

$$\Delta \dot{y}_0 = \alpha [2A(x_0,0)+\dot{\phi}_0]\sin \phi_0.$$

Actually the last relation is there only to determine the sign of the correction to be brought to the initial velocity $\dot{y}_0$. Indeed, the new velocity orthogonal to the axis of symmetry should be computed in norm rather from the energy integral to make sure that the corrector absorbs even the second order variation of the energy constant.

Concerning the above isoenergetic corrector for symmetric periodic orbits, the following remarks are of relevance.

a) The fact that β is equal to zero whenever $Tr(T) \neq 2$, implies that the improved orbit generated by the corrector from a symmetric orbit is also a symmetric orbit.

b) Here no use is made of the fact that, at half the period, the orbit ought to cross orthogonally the axis of symmetry. We know from experience that, unless the orbit is of a simple shape, such a condition makes an uneasy criterion of periodicity. For instance, if the second crossing occurs in the neighborhood of a singularity or close to a curve of zero velocity, the periodic orbit will show intricate loops, which cross several times the axis of symmetry at points which are close to each other and with inclinations that cannot be discriminated distinctly. In these circumstances, it is altogether too tempting to make subjective decisions, thereby running the risk of jumping into

a natural family different from the one to be continued. In our corrector, we converge to the symmetric periodic orbit simply by shifting conveniently the initial position along the axis of symmetry, and by adjusting the period to this translation. The fact that, ultimately when the procedure has converged, the periodic orbit crosses orthogonally the axis of symmetry is not a boundary condition that we impose to decide the time at which we stop the integration, but a consequence of an iteratively convergent adjustment of the initial conditions.

When we use the correction equations (79) for a tangential predictor, we have to assume that

$$n = \alpha n^{I} + \beta n^{II} + n^{III}, \qquad p = \alpha p^{I} + \beta p^{II} + p^{III}$$

and that

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0.$$

Thus we arrive at the formulae

$$\alpha[n^{I}(T/2)-n^{I}(-T/2)] + \beta[n^{II}(T/2)-n^{II}(-T/2)] = -[n^{III}(T/2)-n^{III}(-T/2)],$$
$$\alpha[\dot{n}^{I}(T/2)-\dot{n}^{I}(-T/2)] + \beta[\dot{n}^{II}(T/2)-\dot{n}^{II}(-T/2)] = -[\dot{n}^{III}(T/2)-\dot{n}^{III}(-T/2)], \qquad (84)$$
$$\alpha[p^{I}(T/2)-p^{I}(-T/2)] + \beta[p^{II}(T/2)-p^{II}(-T/2)] + V(T/2)\Delta T = -[p^{III}(T/2)-p^{III}(-T/2)],$$
$$\alpha[\dot{p}^{I}(T/2)-\dot{p}^{I}(-T/2)] + \beta[\dot{p}^{II}(T/2)-\dot{p}^{II}(-T/2)] + \dot{V}(T/2)\Delta T = -[\dot{p}^{III}(T/2)-\dot{p}^{III}(-T/2)].$$

At this stage, we should notice that, along a symmetric orbit, the normal displacement n^{III} is an even function of the time while the resulting tangential displacement p^{III} is an odd function of the time. Therefore,

the predictor formulae (84) become

$$\beta n^{II}(T/2) = 0,$$

$$\alpha \dot{n}^{I}(T/2) = -\dot{n}^{III}(T/2),$$

$$\alpha p^{I}(T/2) + \frac{1}{2} V(T/2)\Delta T = -p^{III}(T/2),$$

$$\beta \dot{p}^{I}(T/2) = 0.$$

(85)

As we have seen above, if we exclude the critical case when $Tr(T) = 2$, we must take $\beta = 0$. In which case, the second of equations (85) yields a unique determination for α while the third equation provides the tangential coefficient $\Delta T = 2dT_0/dC$. Thus in reference to a symmetric periodic orbit,

$$\frac{d}{dC} x_0(C_0) = -\frac{1}{2} \alpha \sin \phi_0,$$

$$\frac{d}{dC} y_0(C_0) = 0,$$

$$\frac{d}{dC} \dot{x}_0(C_0) = 0,$$

(86)

$$\frac{d}{dC} \dot{y}_0(C_0) = -\frac{1}{2} \frac{\sin \phi_0}{V_0} + \frac{1}{2}(2A_0+\dot{\phi}_0)\alpha \sin \phi_0.$$

We should observe that, from a symmetric periodic orbit which is not singular $(Tr(T) \neq 2)$, the tangential predictor generates only symmetric orbits. In other words, in the neighborhood of a symmetric periodic orbit, a natural family consists only of symmetric orbits, and it can be continued by a sequence of nonsymmetric ones only if it goes through a symmetric orbit whose characteristic exponent is zero.

8. ISOENERGETIC VARIATIONS WITH RESPECT TO A PARAMETER

We consider here conservative dynamical systems in which the characteristic functions A and W depend on a parameter ϵ. Let us assume that the right-hand members of the equations (1) verify conditions sufficient for the solutions of (1) to be unique and continuously differentiable in all their arguments for ϵ in a certain neighborhood of ϵ_0. Suppose that $x(t;\epsilon)$, $y(t;\epsilon)$ is a particular solution of (1) for every ϵ in that neighborhood of ϵ_0 and reduces at ϵ_0 to the solution

$$\bar{x}(t), \qquad \bar{y}(t). \tag{87}$$

Then the partial derivatives

$$u(t) = \frac{\partial}{\partial \epsilon} x(t;\epsilon_0), \qquad v(t) = \frac{\partial}{\partial \epsilon} y(t;\epsilon_0)$$

are solutions of the variational equations

$$\ddot{u} = 2A\dot{v} + (W_{xx}+2A_x\dot{y})u + (W_{xy}+2A_y\dot{y})v + (W_{x\epsilon}+2A_\epsilon\dot{y}),$$

$$\ddot{v} = -2A\dot{u} + (W_{xy}-2A_x\dot{x})u + (W_{yy}-2A_y\dot{x})v + (W_{y\epsilon}-2A_\epsilon\dot{x}), \tag{88}$$

to which belongs the variational integral

$$\gamma = W_\epsilon + W_x u + W_y v - \dot{x}\dot{u} - \dot{y}\dot{v}. \tag{89}$$

If a displacement of the solution (87) caused by a variation $\Delta\epsilon$ of the parameter ϵ from its value ϵ_0 for the solution, i.e. a solution of the system (88), is such that the integration constant defined by (89) vanishes, then it is called an *isoenergetic variation of the solution* with respect to the parameter ϵ.

Functions $p(t)$ and $n(t)$ are called respectively tangential and normal displacements of the solution (87) with respect to the parameter ε if the equations (88) possess a solution $u(t), v(t)$ by means of which $p(t)$ and $n(t)$ are representable in the form (15). In particular, if they belong to a displacement $u(t), v(t)$ for which the integration constant γ in (89) vanishes, then they are called isoenergetic.

Proceeding exactly as in paragraph 2, one is able to prove the following proposition:

THEOREM. *A numerical function* $n = n(t)$ *is an isoenergetic normal displacement of (87) caused by a variation of the parameter* ε *if and only if it satisfies the linear differential equation*

$$\ddot{n} + \Theta n = -2W_\varepsilon \frac{A + \dot{\phi}}{V} - 2A_\varepsilon V + W_{y\varepsilon} \cos \phi - W_{x\varepsilon} \sin \phi. \tag{90}$$

In which case, a numerical function $p = p(t)$ *is the corresponding isoenergetic tangential displacement of (87) if and only if it is given by the quadrature*

$$\frac{d}{dt} \left(\frac{p}{V} \right) = 2n \frac{A + \dot{\phi}}{V} + \frac{W_\varepsilon}{V^2}. \tag{91}$$

The homogeneous linear differential equation associated to (90) being Hill's equation, the isoenergetic normal displacements of (87) with respect to the parameter are evidently of the general form

$$n = \alpha n^I + \beta n^{II} + n^\varepsilon,$$

where α and β are constants to be determined by the initial conditions, and n^ε is the particular solution of the nonhomogeneous Hill's equation (90) satisfying the initial condition ;

$$n^\varepsilon(0) = \dot{n}^\varepsilon(0) = 0. \tag{92}$$

9. ISOENERGETIC CHANGE OF THE TIME VARIABLE

The practical importance of the theorem stated in the preceding paragraph arises from its applications to Celestial Mechanics. In a number of problems there occur fixed singularities which are not essential, such as the binary collisions in the Restricted Problem. It is usual to remove them in two steps, by a conformal transformation of the coordinates followed by a change of the time variable in an isoenergetic way. Consequently, in the transformed system, the original constant of energy turns out to have become a parameter, and the variations with respect to C which form the basic ingredient of our tangential predictor are deprived of any meaning for the original problem unless they are isoenergetic in reference to the new problem. To convince ourselves how essential this restriction is, let us recall what is meant by an isoenergetic change of the time in Celestial Dynamics.

Consider a conservative dynamical system described by the Hamiltonian function $\mathfrak{H}$ in the phase space (q,p). The equations of motion

$$\dot{q} = \mathfrak{H}_p, \qquad \dot{p} = -\mathfrak{H}_q \tag{93}$$

admit the integral

$$\mathfrak{H}(q,p) = h \tag{94}$$

Denote by $E(h)$ the manifold defined implicitly by (94) in the phase space (q,p). If only solutions lying on the integral manifold $E(h)$ for a fixed value of the integration constant h are considered, one can produce in a direct manner a rule for the introduction of a new time variable.

Let $G(q,p)$ be a nonzero numerical function defined in the phase space of the system. Along any solution $(q(t),p(t))$ of the equations of motion (93), consider the new time variable s defined by the curvilinear integral

$$s(t) = \int^{t} \frac{du}{G(q(u),p(u))} . \tag{95}$$

Define a conservative Hamiltonian function

$$\mathscr{K}(q,p;h) = (\mathfrak{H}-h)G. \tag{96}$$

Then denoting by a prime differentiation with respect to s, write the canonical equations generated by $\mathscr{K}$,

$$q' = \mathscr{K}_{p}, \qquad p' = -\mathscr{K}_{q}. \tag{97}$$

Denote by $F(h)$ the manifold defined implicitly by the invariant relation

$$\mathscr{K}(q,p;h) = 0 \tag{98}$$

in the phase space.

As a result of a proposition established by Painlevé, only the solutions of the equations (93) in the integral manifold E(h) correspond to solutions of the equations (97), and these lie in the manifold F(h). Conversely, only the solutions of the equations (97) in the manifold F(h) correspond to solutions of the equations (93), and these lie in the manifold E(h).

This duality between the solutions in E(h) and those in F(h) is to be extended to the displacements of the orbits.

Of a displacement $(\delta q, \delta p)$ of an orbit $(q(t), p(t))$ solution of the equations (93), we say that it is $\mathfrak{H}$-isoenergetic if the variation it causes to the Hamiltonian function is zero. In the same way, we call $\mathfrak{K}$-isoenergetic a displacement of an orbit $(q,(s),p(s))$ solution of the equations (97) such that

$$\delta \mathfrak{K} = 0.$$

The orbit obtained by displacing an orbit lying in the manifold F(h) corresponds by Painlevé's duality to an orbit lying in E(h) if and only if the displacement is $\mathfrak{K}$-isoenergetic. In which case, since

$$\delta \mathfrak{K} = G \delta \mathfrak{H} + (\mathfrak{H} - h) \delta G = G \delta \mathfrak{H}$$

in view of the fact that $\mathfrak{H} = h$ along the orbit, to a displacement which is $\mathfrak{K}$-isoenergetic corresponds by duality a displacement which is $\mathfrak{H}$-isoenergetic. Conversely, the orbit obtained by displacing a solution of the equations (93) lying in the manifold E(h) corresponds by Painlevé's duality to a solution of (97) lying in F(h) if and only if the

displacement is $\mathfrak{H}$-isoenergetic; accordingly to a displacement which is

$\mathfrak{H}$-isoenergetic corresponds by duality a displacement which is

$\mathcal{K}$-isoenergetic.

In consequence, when the numerical continuation of a natural

family of periodic orbits for the system described by $\mathfrak{H}$ necessitates

an isoenergetic change of the time, the use of isoenergetic displacements

is no longer optional, but it is a necessity.

In the phase space defined by the Hamiltonian $\mathcal{K}$, the corrector

part of our procedure goes without any modification at all, since it is

based on isoenergetic variations. As for the tangential predictor, we

have to remark that, in the phase space of $\mathcal{K}$, the energy constant h

is to be treated as a parameter. Thus the rates of variation n^{III} and

p^{III} which we can compute in the phase space $\mathfrak{H}$ from a Hill equation

of the type (16) should be computed in the phase space $\mathcal{K}$ in a

$\mathcal{K}$-isoenergetic way from a Hill equation of the type (90).

CONCLUSIONS

The method which we propose here for continuing numerically natural families of periodic orbits is valid for any conservative dynamical system. But to date we have applied it only to the planar Restricted Problem of Three Bodies, whether in barycentric synodical Cartesian coordinates or in regularizing parabolic coordinates. The program has generated with the same ease both symmetric and nonsymmetric orbits. The continuation of a branch through and beyond an ejection orbit has not caused any hardship; the calculation of characteristic exponents for orbits of close approach or for periodic collision courses did not either raise any difficulty. Results in these directions will be reported soon.

The algorithm involves only the first order variations. Consequently, it is unable to treat the singular cases of periodic orbits whose characteristic exponent is equal to zero. These appear as singularities on the analytic manifold. But our procedures are open to the introduction of variations of a higher order, and we are presently engaged in exploring the possibilities offered by the second variations. This way we hope to clarify some of these singular orbits, at least those at which the natural families present a bifurcation of order 2.

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